Thoughts on the reduced Whitehead group of the Iwasawa algebra

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Abstract

Let l be an odd prime and K/k a Galois extension of totally real number fields with Galois group G such that K/k_∞ and $k/\mathbb{Q}$ are finite. We reduce the conjectured triviality of the reduced Whitehead group $SK_1(\mathcal{Q}G)$ of the algebra $\mathcal{Q}G = \mathrm{Quot}(\Lambda G)$ with the Iwasawa algebra $\Lambda G = \mathbb{Z}_l[[G]]$ to the case of pro-l Galois groups G and finite unramified coefficient extensions.

1 Introduction

We fix an an odd prime number l and a Galois extension K/k of totally real fields with Galois group G such that $k/\mathbb{Q}$ and K/k_∞ are finite. As usual, k_∞ denotes the cyclotomic $\mathbb{Z}_l$-extension of k. Next, the Iwasawa algebra $\Lambda G = \mathbb{Z}_l[[G]] = \varprojlim_{N \lhd G} \mathbb{Z}_l[G/N]$, where N runs through the open normal subgroups of G, denotes the completed group ring of G over $\mathbb{Z}_l$ and $\mathcal{Q}G = \mathrm{Quot}(\mathbb{Z}_l[[G]])$ is its total ring of fractions with respect to all central non-zero divisors. Let $K_0T(\Lambda G)$ be the Grothendieck group of the category of finitely generated torsion ΛG-modules of finite projective dimension. Then, the localization sequence of K-theory

$$\to K_1(\Lambda G) \to K_1(\mathcal{Q}G) \xrightarrow{\partial} K_0T(\Lambda G) \to$$

is exact. $\mathcal{Q}G$ finds its way into non-commutative Iwasawa theory via this localization sequence and a determinant map

$$\mathrm{Det} : K_1(\mathcal{Q}G) \to \mathrm{Hom}(R_l G, (\mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma_k)^\times),$$

where $\mathbb{Q}_l^c$ is a fixed algebraic closure of $\mathbb{Q}_l$, the $\mathbb{Z}_l$-span of the irreducible $\mathbb{Q}_l^c$-characters of G with open kernel is named $R_l G$ and $\Gamma_k = G(k_\infty/k)$. This determinant is the translation of the reduced norm $\mathrm{nr} : K_1(\mathcal{Q}G) \to Z(\mathcal{Q}G)^\times$ to Hom groups, where $Z(\mathcal{Q}G)$ is the centre of $\mathcal{Q}G$. We refer to [6] for a precise definition of Det.

As in the classical case of Iwasawa, Ritter and Weiss link a K-theoretic substitute $\mho \in K_0T(\Lambda G)$ of the Iwasawa module X to the Iwasawa L-function which is derived from the S-truncated Artin L-function for a finite set S of places of k containing all archimedian ones and those which ramify in K (see e.g. [6]). This Iwasawa L-function lies in the upper Hom-group.

With this, the main conjecture of equivariant Iwasawa theory says

There exists a unique element $\Theta \in K_1(\mathcal{Q}G)$ *s.t.* $\mathrm{Det}(\Theta) = L$. *Moreover* $\partial(\Theta) = \mho$.

The uniqueness of Θ would follow from the conjecture by Suslin that the reduced Whitehead group $SK_1(A)$ is trivial for central simple algebras A over fields with cohomological dimension ≤ 3 (see [12] for this conjecture of Suslin). In [4, Thm 5.1], it is shown that this conjecture can be applied to our algebra $\mathcal{Q}G$.

Recently (compare [9]), Ritter and Weiss gave a complete proof of this main conjecture up to its uniqueness statement whenever Iwasawa's μ-invariant vanishes. In [3], Kakde also gave a proof. In fact, he does not restrict to 1-dimensional l-adic Lie groups as Ritter and Weiss do but gives a proof for higher dimensional admissible l-adic Lie groups. Yet, he does not consider the full ring of fractions $\mathcal{Q}G$ but the localization $(\Lambda G)_S$ by the canonical Ore set S and proves uniqueness up to the quotient of $K_1((\Lambda G)_S)$ by the image of $SK_1(\Lambda G))$. Thus, the question whether Θ is unique in $K_1(\mathcal{Q}G)$ is still open.

In this paper, we reduce the Suslin conjecture for our Iwasawa algebra $\mathcal{Q}G$ for profinite Galois groups G to the conjecture for $N \otimes_{\mathbb{Q}_l} \mathcal{Q}U$ for pro-l groups U and finite unramified extensions N of $\mathbb{Q}_l$. Therefore, the proof of the uniqueness statement of the main conjecture is completely reduced to pro-l groups provided that the studied objects are unaffected by passing to finite unramified extensions of $\mathbb{Q}_l$.

This paper contains some of the results of my PhD thesis. I would like to thank my supervisor Jürgen Ritter for his aid, encouragement and patience during my work on this paper.

2 Recollections

First, we recall some facts on the structure of $\mathcal{Q}G$ and formulate the Suslin conjecture.

We keep the notation of the introduction, in particular we fix an odd prime l and a Galois extension K/k of totally real fields with Galois group G such that $k/\mathbb{Q}$ and K/k_∞ are finite.

First, G splits (see [6, p. 551]): $G = H \rtimes \Gamma$ with $H = G(K/k_\infty)$ and $\Gamma = \langle\gamma\rangle \cong G(k_\infty/k) \cong \mathbb{Z}_l$. Thus, for a central subgroup $\Gamma^{l^m} =: \Gamma_0$ we get

$$\mathcal{Q}G = \bigoplus_{i=0}^{l^m-1} (\mathcal{Q}\Gamma_0)[H]\gamma^i.$$

This algebra is a finite dimensional $\mathcal{Q}\Gamma_0$-algebra; in fact, it is a semisimple algebra, since the Jacobson radical is trivial by [6, p. 553]. Now, let $\chi \in R_lG$ be an irreducible $\mathbb{Q}_l^c$-character of G with open kernel. Note that it is sufficient to regard the finite set of irreducible characters of G/Γ_0 because, by inflation and twist with irreducible characters ρ which fulfil $\mathrm{res}_G^H \rho = 1$, every irreducible $\chi \in R_lG$ can be obtained from this set. These characters ρ will be called of type W. Because G is an l-group with $l \neq 2$, this implies that χ has a representation over $\mathbb{Q}_l(\chi)$ by [10].

Furthermore, with η an absolutely irreducible constituent of $\mathrm{res}^H_G(\chi)$, we define

$$St(\eta) := \{g \in G : \eta^g = \eta\}, \ w_\chi := [G : St(\eta)]$$

and

$$e(\eta) := \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1})h.$$

Ritter and Weiss showed in [6] that

$$e_\chi := \sum_{\eta | \mathrm{res}^H_G \chi} e(\eta)$$

is a central primitive idempotent in $\mathcal{Q}^c G$, that every central primitive idempotent is of the form e_χ and that two central primitive idempotents e_{χ_1} and e_{χ_2} coincide if and only if $\chi_1 = \chi_2 \otimes \rho$ for a character ρ of type W.

In the special case of a pro-l group G, the structure of $\mathcal{Q}G$ is completely known by [4]:

Lemma 1 *Let now G be a pro-l group and let W' be the simple component of $\mathcal{Q}G$ corresponding to the irreducible character $\chi \in R_l G$. We moreover choose an absolutely irreducible constituent η of $\mathrm{res}^H_G(\chi)$. Then,*

(i)

$$W' \cong \bigoplus_{i=0}^{l^m-1} \left(\bigoplus_{j=0}^{v_\chi - 1} (\mathbb{Q}_l(\eta) \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma_0)_{\eta(1)\times\eta(1)} \right) \gamma^i$$

for $v_\chi := min\{0 \le j \le w_\chi - 1 : \eta^{\gamma^j} = \eta^\sigma$ for some $\sigma \in G(\mathbb{Q}_l(\eta)/\mathbb{Q}_l)\}$,

(ii) W' has centre

$$Z(W') \cong L \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma^{w_\chi}$$

with $L = \mathbb{Q}_l(\eta)^{G_0}$ and $G_0 = \{\sigma \in G(\mathbb{Q}_l(\eta)/\mathbb{Q}_l) : \eta^\sigma = \eta^{\gamma^j}$ for a $0 \le j \le w_\chi - 1\}$. Moreover, $G_0 =: \langle \sigma_{v_\chi} \rangle$ is a cyclic group of order $\frac{w_\chi}{v_\chi}$.

(iii) $Z(W')$ has cohomological dimension $\mathrm{cd}(Z(A)) = 3$,

(iv) $\dim_{Z(W')} W' = \chi(1)^2$,

(v) W' is split by $\mathbb{Q}_l(\eta) \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma^{w_\chi}$,

(vi) W' has Schur index $s_D = w_\chi / v_\chi$ and

(vii) $W' \cong D_{n \times n}$ with $n = \chi(1)/s_D$ and the skew field D is cyclic:

$$D \cong \bigoplus_{i=0}^{w_\chi/v_\chi - 1} (\mathbb{Q}_l(\eta) \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma^{w_\chi})\gamma^{v_\chi i} =: (\mathbb{Q}_l(\eta) \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma^{w_\chi} / L \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma^{w_\chi}, \sigma_{v_\chi}, \gamma^{w_\chi}).$$

Proof: Statements (i) and (ii) can be found in [4, Prop 1], (iii) is [4, Thm 2] and [4, Thm 1] contains (iv) to (vii). □

Because H is a finite l-group, $\mathbb{Q}_l(\eta)$ is generated by a primitive l-power root of unity. Therefore, $L = \mathbb{Q}_l(\eta)^{G_0} \subseteq \mathbb{Q}_l(\eta)$ also is, i.e. we can fix a primitive l-power root of unity ξ s.t.

$$L = \mathbb{Q}_l(\xi).$$

We now focus on the Suslin conjecture.

Definition 1 *(i) For a field F and a central simple F-algebra A of finite degree $[A : F]$, let $\mathrm{nr}_{A/K}$ denote the reduced norm from A to K. The group*

$$SK_1(A) := \ker(\mathrm{nr}_{A/F})/[A^\times, A^\times]$$

is called the reduced Whitehead group of A.

(ii) For a semisimple algebra $A = \bigoplus_i A_i$ of finite degree with simple components A_i, we set

$$SK_1(A) := \bigoplus_i SK_1(A_i)$$

for the reduced Whitehead group of A.

The reduced norm $\mathrm{nr}_{A/F}$ on A induces a homomorphism on $K_1(A)$, which we will call reduced norm, too. We state the following well-known results without proof.

Lemma 2 *(i) Let A be a central simple F-algebra of finite degree. Then*

$$SK_1(A) = \ker(\mathrm{nr}_{A/F} : K_1(A) \to K_1(F)).$$

(ii) Let $A \cong D_{n\times n}$ be the full matrix ring of finite degree over a skew field D. Then

$$SK_1(A) = SK_1(D).$$

(iii) For a field F, we have

$$SK_1(F) = 1.$$

For further details, see e.g. [2, Part III].

Remark 1 *The determonant map* Det *in the main conjecture of equivariant Iwasawa theory is the translation of the reduced norm to the language of Hom-groups. For a detailed definition of this Det, we refer to [6, p. 558].*

We are now ready to state the

Conjecture *Let F be a field with cohomological dimension* $\mathrm{cd}(F) \leq 3$ *and A a central simple F-algebra of finite degree $[A : F]$. Then*

$$SK_1(A) = 1.$$

In the following, we will call this Suslin's conjecture, although this is not literally Suslin's formulation. But in the case of a field of cohomological dimension less than or equal to 3, this is exactly the statement of his conjecture. For details, we refer to [12].

The centres of the Wedderburn components of $\mathcal{Q}G$, i.e. the simple components W', are of cohomological dimension 3 for pro-l groups G by Lemma 1. As we will see in this paper, this is the crucial case for the triviality of $SK_1(\mathcal{Q}G)$.

Next, we list the cases for $\mathcal{Q}G$ which are known to have trivial reduced Whitehead group:

Lemma 3 *Let G be as above. Then, $SK_1(\mathcal{Q}G) = 1$ in the following cases:*

(i) $G = H \times \Gamma$ *is a direct product with H an l-group or of order prime to l*[1].

(ii) G *is a pro-l group G with abelian subgroup of index l.*

(iii) $G = H \rtimes \Gamma$, *where H is a finite group of order prime to l.*

Proof: For l-groups H, Roquette has shown in [10] that $\mathbb{Q}_l[H]$ is the direct sum of some matrix rings over fields. Therefore, $\mathcal{Q}G = (\mathcal{Q}\Gamma)[H] = \mathcal{Q}\Gamma \otimes_{\mathbb{Q}_l} \mathbb{Q}_l[H]$ also is a direct sum of matrix rings over fields and thus $SK_1(\mathcal{Q}G) = 1$ by Lemma 2. This shows the first case of (i). The latter statement of (i) is a special case of (iii).

(ii) is shown in [8, p. 118].

(iii) can be found in [7, Example 2, p. 169]. □

3 Reduction of the reduced Whitehead group $SK_1(\mathcal{Q}G)$ to the pro-l case

As main ingredient of our reduction, we cite the following lemma (see [7, Cor, p. 167]). For this, recall that, for a prime number $q \neq l$, G is a $\mathbb{Q}_l$-q-elementary group if $G = H \times \Gamma$ with Γ a central open subgroup of G isomorphic to $G(k_\infty/k)$ and H a finite $\mathbb{Q}_l$-q-elementary group; i.e. $H = \langle s \rangle \rtimes H_q$ is the semidirect product of a cyclic group $\langle s \rangle$ of order prime to q and a q-group H_q whose action on $\langle s \rangle$ induces a homomorphism $H_q \to G(\mathbb{Q}_l(\zeta)/\mathbb{Q}_l)$. Here, ζ is a primitive root

[1] We will see in subsection 3.2 that the restrictions on H are not necessary.

of unity of order $|\langle s\rangle|$. For $q = l$, the group G is called $\mathbb{Q}_l$-l-elementary if $G = \langle s\rangle \rtimes U$ is the semidirect product of a finite cyclic group $\langle s\rangle$ of order prime to l and an open pro-l subgroup U whose action on $\langle s\rangle$ induces a homomorphism $U \to G(\mathbb{Q}_l(\zeta)/\mathbb{Q}_l)$ with again ζ a primitive root of unity of order $|\langle s\rangle|$.

Lemma 4 (Ritter, Weiss) *Let K/k be a Galois extension of totally real fields with Galois group G such that K/k_∞ and $k/\mathbb{Q}$ are finite. Then, $SK_1(\mathcal{Q}G) = 1$ if $SK_1(\mathcal{Q}G') = 1$ for all open $\mathbb{Q}_l$-q-elementary subgroups G' of G and all prime numbers q (q might be equal to l).*

Thus, we have to compute $SK_1(\mathcal{Q}G)$ for $\mathbb{Q}_l$-q-elementary groups G with q running through the set of all prime numbers.

3.1 $\mathbb{Q}_l$-l-elementary groups G

We begin with the case $q = l$, i.e. $G = \langle s\rangle \rtimes U$ with a finite cyclic group $\langle s\rangle$ of order prime to l and U an open pro-l subgroup.

We fix a finite set $\{\beta_i\}$ of representatives of the $G(\mathbb{Q}_l^c/\mathbb{Q}_l)$-orbits of the irreducible $\mathbb{Q}_l^c$-characters of $\langle s\rangle$. Let also ζ_i denote a fixed primitive l-prime root of unity with $\beta_i(s) = \zeta_i$.

Let $U_i := \{u \in U : \beta_i^u = \beta_i\}$ denote the stabilizer group of β_i. Clearly, $U_i \lhd U$ and $A_i := U/U_i \leq G(\mathbb{Q}_l(\beta_i)/\mathbb{Q}_l) = G(\mathbb{Q}_l(\zeta_i)/\mathbb{Q}_l)$. Thus, A_i is cyclic because $\mathbb{Q}_l(\beta_i) = \mathbb{Q}_l(\zeta_i)$ is unramified over $\mathbb{Q}_l$. We fix a representative $x_i \in U$ with $\langle\overline{x_i}\rangle = U/U_i = A_i$. Then, $\overline{x_i}$ maps to some τ_i under the injection $U/U_i \rightarrowtail G(\mathbb{Q}_l(\beta_i)/\mathbb{Q}_l) = G(\mathbb{Q}_l(\zeta_i)/\mathbb{Q}_l)$ and therefore the order of $\overline{x_i}$ clearly is a power of l, say $l^n := |U/U_i|$. Although n depends on i, we omit this in the notation. Moreover, for the sake of brevity, we set $x := x_i$ and $\tau := \tau_i$, but still keep in mind the underlying β_i. Finally, we set $G_i := \langle s\rangle \rtimes U_i$.

We next read the structure of $\mathcal{Q}G$ in these terms. For this, recall that the

$$e_i := \frac{1}{|\langle s\rangle|} \sum_{\nu \bmod |\langle s\rangle|} \mathrm{tr}_{\mathbb{Q}_l(\zeta_i)/\mathbb{Q}_l}(\zeta_i(s^{-\nu}))s^\nu \in \mathbb{Z}_l\langle s\rangle$$

are the primitive central idempotents of the group algebra $\mathbb{Q}_l\langle s\rangle$, and furthermore they are central idempotents of $\mathcal{Q}G$. Because the e_i are orthogonal in $\mathbb{Q}_l\langle s\rangle$, we have $e_ie_j = 0$ for $i \neq j$ in $\mathcal{Q}G$, too. Therefore, we conclude $\bigoplus_i e_i\mathcal{Q}G \subseteq \mathcal{Q}G$. For

$$\mathcal{Q}G = \bigoplus_i e_i\mathcal{Q}G,$$

it remains to show the other inclusion $\mathcal{Q}G \subseteq \bigoplus_i e_i\mathcal{Q}G$. We use that $\sum_i e_i = 1$ is true in $\mathbb{Q}_l\langle s\rangle$ and therefore it is true in $\mathcal{Q}G$, too. Thus, $\mathcal{Q}G = 1 \cdot \mathcal{Q}G \subseteq \bigoplus_i e_i\mathcal{Q}G$. We are now ready to state

Lemma 5 *With the above notations, we have:*

(i) $e_i\mathcal{Q}G_i \cong \mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$,

(ii) $e_i \mathcal{Q}G \cong \bigoplus_{j=0}^{l^n-1} (\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i)x^j$,
where x acts on U_i by conjugation and on $\mathbb{Q}_l(\zeta_i)$ via τ.

Proof: (i) is stated in [7, p. 160] and (ii) follows immediately by (i) and the definition of U_i. □

To point out the importance of the operation of x, we will also use the notation

$$(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle := \bigoplus_{j=0}^{l^n-1} (\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i)x^j.$$

Proposition 1 *With the above notations, the following are equivalent:*

(i) $SK_1(\mathcal{Q}G) = 1$.

(ii) $SK_1(e_i\mathcal{Q}G) = SK_1((\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle) = 1$ *for all characters β_i of $\langle s \rangle$.*

Proof: This follows immediately by $\mathcal{Q}G = \bigoplus_i e_i\mathcal{Q}G$ and Lemma 5. □

As the structure of $\mathcal{Q}U_i$ is well known from Section 2, we now examine $(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle$. Because $(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle$ is isomorphic to $e_i\mathcal{Q}G$, this algebra is semisimple.

Let W' be the Wedderburn component, i.e. the simple component, of $\mathcal{Q}U_i$ corresponding to $\chi \in R_l U_i$ and set

$$W = \mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} W' = (\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} Z(W')) \otimes_{Z(W')} W' \subseteq \mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i.$$

As $\mathbb{Q}_l(\zeta_i)$ and $F' := Z(W') = L \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma^{w_\chi}$ are linearly disjoint over $\mathbb{Q}_l$, the tensor product $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} F'$ is a field and thus W is still a simple algebra and therefore a Wedderburn component of $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$ with centre $F := Z(W) = \mathbb{Q}_l(\zeta_i) \otimes F'$.

Then, x acts on W as it acts on $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$. This action fixes the algebra $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$ as a whole, but might not fix W. If $W^x \neq W$, then W^x is another Wedderburn component of $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$ by the following: W^x is a two-sided ideal of $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$ because W is a two-sided ideal of $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$. Furthermore, it has centre F^x with $F = Z(W)$. As seen above, $F = \mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} L \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma^{w_\chi}$ is a field and therefore F^x is a field, too. But as a semisimple algebra with a field as centre, W^x is already a simple algebra. Thus, x permutes the Wedderburn components of $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$ and $W^x \cdot W = 0$ if $W^x \neq W$ because of the orthogonality of Wedderburn components.

Note that the minimal j, such that $W^{x^j} = W$, is an l-power because this is the length of the orbit of W in the set of Wedderburn components of $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$ under the action of $\langle \overline{x} \rangle$.

Proposition 2 *Let W be a simple component of $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$ with centre F. Set $0 \leq d \leq n$ to be minimal such that $W^{x^{l^d}} = W$. Then,*

$$\tilde{W} := \bigoplus_{j=0}^{l^d-1} (W^{x^j} \oplus W^{x^j}x \oplus ... \oplus W^{x^j}x^{l^n-1})$$

is a simple component of $(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle$ *with centre* $Z(\tilde{W}) = F^{\langle x^{l^d} \rangle} =: E$. *Furthermore,* $\tilde{W}$ *is the full matrix ring*

$$\tilde{W} = V_{l^d \times l^d} \quad \textit{with} \quad V := W \oplus W x^{l^d} \oplus ... \oplus W x^{l^d (l^{n-d}-1)}.$$

Proof: We set $y := x^{l^d}$ and $m := n - d$, i.e. $y^{l^m} = x^{l^n} \in U_i$.

First, $\tilde{W}$ is a two-sided ideal of $(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle$; for this, we only have to check that it is closed under multiplication with x, which is obvious. Thus it is the direct sum of some Wedderburn components.

Next, we show that the centre of $\tilde{W}$ is a field, which automatically implies that $\tilde{W}$ is a simple algebra. We start with the computation of the centre $Z(V)$ of V. Here, we do not consider the trivial case $d = n$, i.e. $V = W$ and therefore $Z(V) = Z(W) = F$ is a field.

We assume $0 \leq d < n$ and take an element $z = w_0 + w_1 y + ... + w_{l^m-1} y^{l^m-1} \in Z(V)$. For any $w \in W \subseteq V$, we see

$$\begin{aligned} zw &= w_0 w + w_1 w^{y^{-1}} y + ... + w_{l^m-1} w^{y^{-(l^m-1)}} y^{l^m-1}, \\ wz &= w w_0 + w w_1 y + ... + w w_{l^m-1} y^{l^m-1}. \end{aligned}$$

Because $zw = wz$, we conclude that $w_0 \in Z(W) = F$ and

$$w_1 w^{y^{-1}} = w w_1 \ , \ ... \ , \ w_{l^m-1} w^{y^{-(l^m-1)}} = w w_{l^m-1}. \tag{1}$$

Assume for the moment that $w \in Z(W) = F$. Then, (1) implies

$$w_1 w^{y^{-1}} = w_1 w \ , \ ... \ , \ w_{l^m-1} w^{y^{-(l^m-1)}} = w_{l^m-1} w.$$

But as $F = \mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} L \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma^{w_\chi}$, we can specialize to $w = \zeta_i$. By definition, y does not act trivially on ζ_i (otherwise $y \in U_i$) and thus $w_1 = ... = w_{l^m-1} = 0$.

Moreover, z fulfils $yz = zy$. As we have already seen that $z = w_0 \in F$, this implies that $z \in F^{\langle y \rangle}$. Thus $Z(V) \subseteq F^{\langle y \rangle}$. Because the other inclusion $Z(V) \supseteq F^{\langle y \rangle}$ is trivially true, we finally conclude

$$Z(V) = F^{\langle y \rangle}.$$

Now, we are ready to show that $Z(\tilde{W}) = Z(V) = F^{\langle y \rangle}$. For the rest of the proof, we will again allow the trivial case, i.e. $0 \leq d \leq n$. We use the relation

$$\begin{aligned} \tilde{W} &= \bigoplus_{j=0}^{l^d-1} (W^{x^j} \oplus W^{x^j} x \oplus ... \oplus W^{x^j} x^{l^n-1}) \\ &= \bigoplus_{j=0}^{l^d-1} (V^{x^j} \oplus V^{x^j} x \oplus ... \oplus V^{x^j} x^{l^d-1}). \end{aligned}$$

Let $0 \leq j \leq l^d - 1$. Because $W^{x^j} \neq W$, we have seen $W^{x^j} \cdot W = 0$ and therefore $V^{x^j} \cdot V = 0$. We choose $z = \sum_{i,j=0}^{l^d-1} v_{ij}^{x^j} x^i \in Z(\tilde{W})$, and $v, v' \in V$. Then

$$zv = \sum_{i,j} v_{ij}^{x^j} v^{x^{-i}} x^i = v_{00}v + \sum_{i>0} (v_{i,l^d-i} v^{x^{-l^d}})^{x^{l^d-i}} x^i \in V \oplus \bigoplus_{i=1}^{l^d-1} V^{x^{l^d}-i} x^i,$$
$$vz = \sum_{i,j} v v_{ij}^{x^j} x^i = \sum_i v v_{i0} x^i = v v_{00} + \sum_{i>0} v v_{i0} x^i \in V \oplus \bigoplus_{i=1}^{l^d-1} V x^i.$$

Thus, $v_{00} \in Z(V)$ and the orthogonality of the V^{x^j} implies $v_{i,l^d-i} = 0 = v_{i0}$ for all $i > 0$. Next,

$$zv^x = \sum_{i,j} v_{ij}^{x^j} v^{x^{-i+1}} x^i = (v_{01}v)^x + v_{10} v x + \sum_{i>1} (v_{i,l^d-i+1} v^{x^{-l^d}})^{x^{l^d-i+1}} x^i,$$
$$v^x z = \sum_{i,j} v^x v_{ij}^{x^j} x^i = \sum_i (v v_{i1})^x x^i = (v v_{01})^x + (v v_{11})^x x + \sum_{i>1} (v v_{i1})^x x^i.$$

Thus, $v_{01} \in Z(V)$, $v_{10} = 0 = v_{11}$ and $v_{i,l^d-i+1} = 0 = v_{i1}$ for all $i > 1$. Analogous computations for $zv^{x^\nu} = v^{x^\nu} z$ finally lead to

$$z = v_0 + v_1^x + ... + v_{l^d-1}^{x^{l^d-1}}$$

with $v_i \in Z(V)$. We apply this together with the orthogonality and compute

$$z(v + v'x^i) = v_0 v + v_0 v' x,$$
$$(v + v'x^i)z = v v_0 + v' v_i x = v_0 v + v_i v' x$$

with $1 \leq i \leq l^d - 1$. Thus, $v_i = v_0$ for all $1 \leq i \leq l^d - 1$.

Therefore, we have achieved $z \in \{\sum_{j=0}^{l^d-1} v^{x^j} : v \in Z(V)\} \cong Z(V)$, i.e. $Z(\tilde{W}) \subseteq Z(V)$. For the other inclusion, it remains to show that elements of $\{\sum_{j=0}^{l^d-1} v^{x^j} : v \in Z(V)\}$ are already central in $\tilde{W}$. For this, we only have to check that $\sum_{j=0}^{l^d-1} v^{x^j}$ commutes with x for every $v \in Z(V) = F^{\langle x^{l^d} \rangle}$:

$$\left(\sum_{j=0}^{l^d-1} v^{x^j} \right)^x = \sum_{j=0}^{l^d-1} v^{x^{j+1}} = \sum_{j=1}^{l^d-1} v^{x^j} + v^{x^{l^d}} = \sum_{j=1}^{l^d-1} v^{x^j} + v = \sum_{j=0}^{l^d-1} v^{x^j}.$$

Hence, $Z(\tilde{W}) = Z(V)$ is true. This moreover shows that $\tilde{W}$ is a Wedderburn component of $(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle$.

It remains to show that $\tilde{W}$ is the claimed matrix ring. First, if $\tilde{W}$ is a matrix ring over V, then the dimension is clear because

$$\begin{aligned} \dim_{Z(\tilde{W})} \tilde{W} = \dim_{Z(V)} \tilde{W} &= \dim_{Z(V)} \bigoplus_{j=0}^{l^d-1} (V^{x^j} \oplus V^{x^j} x \oplus ... \oplus V^{x^j} x^{l^d-1}) \\ &= l^{2d} \dim_{Z(V)} V \end{aligned}$$

and therefore $\dim_V \tilde{W} = \dim_{Z(V)} \tilde{W} / \dim_{Z(V)} V = l^{2d}$.

Both V and $\tilde{W}$ are central simple algebras over $F^{\langle y\rangle}$. We show that $V \sim \tilde{W}$ in $\mathrm{Br}(F^{\langle y\rangle})$, i.e. that V and $\tilde{W}$ are full matrix rings over the same skew field D of centre $F^{\langle y\rangle}$.

For the computation of the skew field D in $\tilde{W}$, we recall the fact that there exists a primitive idempotent ε of $\tilde{W}$ such that $D \cong \varepsilon\tilde{W}\varepsilon$ and $\tilde{W} \cong B^n$ for a minimal right ideal $B = \varepsilon\tilde{W}$ of $\tilde{W}$. Analogously, there exists a primitive idempotent $\varepsilon_V \in V$ with $\varepsilon_V V \varepsilon_V \cong D_V$ a skew field and $S = \varepsilon_V V$ a minimal right ideal of V. Then, we get $\varepsilon_V \tilde{W} \varepsilon_V = \varepsilon_V V \varepsilon_V$ because for $\sum_{i,j=0}^{l^d-1} v_{ij}^{x^j} x^i \in \tilde{W}$, we achieve

$$\begin{aligned}
\varepsilon_V \cdot \left(\sum_{i,j=0}^{l^d-1} v_{ij}^{x^j} x^i \right) \cdot \varepsilon_V &= \varepsilon_V \cdot \left(\sum_{i,j=0}^{l^d-1} v_{ij}^{x^j} \varepsilon_V^{x^{-i}} x^i \right) \\
&\overset{1}{=} \varepsilon_V v_{00} \varepsilon_V + \varepsilon_V \cdot \left(\sum_{i=1}^{l^d-1} (v_{i,l^d-i}^{x^{l^d}})^{x^{-i}} \varepsilon_V^{x^{-i}} x^i \right) \\
&= \varepsilon_V v_{00} \varepsilon_V + \left(\sum_{i=1}^{l^d-1} \varepsilon_V (v_{i,l^d-i}^{x^{l^d}})^{x^{-i}} x^i \right) \cdot \varepsilon_V \\
&\overset{2}{=} \varepsilon_V v_{00} \varepsilon_V .
\end{aligned}$$

For $\overset{1}{=}$ and $\overset{2}{=}$, we have again used $V^{x^j} \cdot V = 0$ for $1 \leq j \leq l^d - 1$.

Next, as $\varepsilon_V \tilde{W}$ is a right ideal of $\tilde{W}$, there exists a $0 < r \in \mathbb{N}$ with $B^r \cong \varepsilon_V \cdot \tilde{W}$ for the minimal right ideal B. Because $\mathrm{End}_{\tilde{W}}(B) \cong D$ is the skew field lying in $\tilde{W}$, we get

$$\begin{aligned}
D_V \cong \varepsilon_V V \varepsilon_V = \varepsilon_V \tilde{W} \varepsilon_V &\cong \mathrm{End}_{\tilde{W}}(\varepsilon_V \tilde{W}) \\
&\cong \mathrm{End}_{\tilde{W}}(B^r) \cong \mathrm{End}_{\tilde{W}}(B)_{r\times r} \cong D_{r\times r}.
\end{aligned}$$

This forces $r = 1$ (because, for example, D_V does not have zero divisors whereas $D_{r\times r}$ has for $r > 1$). Thus, the underlying skew fields of V and $\tilde{W}$ are equal, i.e. $V \sim \tilde{W}$ in $\mathrm{Br}(E)$. By $V \subseteq \tilde{W}$, this implies the claim and concludes the proof. □

Remark 2 *The $\tilde{W}$ in Proposition 2 exhaust all simple components of $(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x\rangle$ because*

$$\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i = \bigoplus_W W$$

and therefore

$$(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x\rangle = \left(\bigoplus_W W \right) \star \langle x\rangle = \bigoplus_{\tilde{W}} \tilde{W}.$$

Corollary 1 *With the notations of Proposition 2, we have*

$$SK_1(\tilde{W}) = SK_1(V).$$

Proof: This is obvious, as the reduced Whitehead group of a simple algebra only depends on the underlying skew field but not on the matrix degree. $\square$

Since the Wedderburn components $\tilde{W}$ are classified now, we can start our study of $SK_1(\tilde{W})$.

Theorem 1 *Let $G = \langle s \rangle \rtimes U$ be a $\mathbb{Q}_l$-l-elementary group with a finite cyclic group $\langle s \rangle$ of order prime to l and U an open pro-l subgroup. Assume that $SK_1(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) = 1$ for all i. Then*

$$SK_1(\mathcal{Q}G) = 1.$$

The proof of this theorem depends on the number $l^d = \min\{1 \leq j \leq l^n : \; W^{x^j} = W\}$.

3.1.1 The case $d = n$

First, let $d = n$, i.e. $W^{x^j} \neq W$ for all $1 \leq j \leq l^n - 1$. Thus,

$$\tilde{W} = \bigoplus_{j=0}^{l^n-1} (W^{x^j} \oplus ... \oplus W^{x^j} x^{l^n-1}), \quad V = W.$$

Then, Proposition 2 implies that $\tilde{W} = V_{l^n \times l^n} = W_{l^n \times l^n}$. Furthermore, both W and $\tilde{W}$ have centre $Z(W) = Z(\tilde{W}) = F$. (Observe that F commutes with x^{l^n} because $x^{l^n} \in U_i$ and $F = Z(W)$ for a Wedderburn component W of $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$.) By Corollary 1, together with the precondition that $SK_1(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) = 1$ and in particular $SK_1(W) = 1$, this also implies

Proposition 3 *With the above notations, assume $W^{x^j} \neq W$ for all $1 \leq j \leq l^n - 1$ and $SK_1(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) = 1$ for all i. Then*

$$SK_1(\tilde{W}) = SK_1(W) = 1.$$

$\square$

3.1.2 The case $d = 0$

Next, we consider $d = 0$, i.e. $W^x = W$. Thus,

$$\tilde{W} = \bigoplus_{j=0}^{l^n-1} W x^j, \quad V = \tilde{W}.$$

This time, $Z(\tilde{W}) = F^{\langle x \rangle} = E$ with $[F : E] = l^n$ and $G(F/E) = \langle \sigma \rangle$, where σ is induced by the conjugation by x. Note that

$$\langle \sigma \rangle \cong \langle \overline{x} \rangle \cong \langle \tau \rangle \subseteq G(\mathbb{Q}_l(\zeta_i)/\mathbb{Q}_l)$$

with τ as above induced by the action of x on $\mathbb{Q}_l(\zeta_i)$.

First, we give a brief outline of the proof of $SK_1(\tilde{W}) = SK_1(V) = 1$.

Step 1 $F \otimes_E V \cong W_{l^n \times l^n} \subseteq V_{l^n \times l^n}$.

Step 2 *There exists a* $w \in W_{l^n \times l^n}$ *such that the conjugation by* $w^{-1}x$ *is the automorphism* $C_{w^{-1}x} = \sigma \otimes 1$ *on* $W_{l^n \times l^n}$. *It is of order* l^n *and* $(w^{-1}x)^{l^n} \in Z(W_{l^n \times l^n}) = F$.

Step 3 $(w^{-1}x)^{l^n} = 1$ *and therefore* $A = (F/E, \sigma, (w^{-1}x)^{l^n}) \subseteq V_{l^n \times l^n}$ *is a central simple split* E*-algebra.*

Step 4 $Z_{V_{l^n \times l^n}}(A) = (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$.

Step 5 $V_{l^n \times l^n} \cong A \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$.

Step 6 $SK_1(V) = SK_1((W_{l^n \times l^n})^{\langle w^{-1}x \rangle}) = 1$.

We now start with the sketched computations. We can read V as free left W-module of rank l^n with basis $1, x, ..., x^{l^n-1}$, i.e. $V = \bigoplus_{j=0}^{l^n-1} Wx^j$. This allows us to formulate

Lemma 6 *With the above notations, we have an isomorphism*

$$F \otimes_E V \xrightarrow{\cong} W_{l^n \times l^n} = \mathrm{Hom}_W(V, V), \quad f \otimes v \mapsto l_f \circ r_v,$$

where l_f *resp.* r_v *denotes the left resp. right multiplication with* $f \in F$ *resp.* $v \in V$.

Remark 3 *In particular,* $f \otimes 1 \in F \otimes_E V$ *maps to the diagonal matrix* $f \cdot \mathbf{1}$ *with* $\mathbf{1}$ *the unit matrix.*

Proof: $F \otimes_E V$ and $W_{l^n \times l^n}$ are isomorphic by [5, Cor 7.14], we only have to substitute K by E, A by V and B by F. Then, we get $r = l^n$ and the centralizer $B' = Z_V(F) = W$ implies $F \otimes_E V = F^{op} \otimes_E V \cong W_{l^n \times l^n}$.

We will call the stated isomorphism φ for the moment. We read the actions of the W-endomorphisms of V by the right to ensure that φ is compatible with multiplication. For this, take $f,\ f' \in F$ and $v,\ v',\ a \in V$. Then the commutativity of F yields

$$\begin{aligned}(a)(\varphi(f \otimes v) \circ \varphi(f' \otimes v')) &= (a)((l_f \circ r_v) \circ (l_{f'} \circ r_{v'})) \\ &= f'favv' = ff'avv' \\ &= (a)(l_{ff'} \circ r_{vv'}) = (a)\varphi(ff' \otimes vv') \\ &= (a)\varphi((f \otimes v)(f' \otimes v')).\end{aligned}$$

It now easily follows that φ is a homomorphism of E-algebras.

$F \otimes_E V$ is simple because V is a central simple E-algebra. Thus, $\varphi \neq 0$ implies that φ is injective. By dimension comparison, it is surjective as well. $\square$

Next, we construct the automorphism $C_{w^{-1}x}$ on $W_{l^n\times l^n}$. On the one hand, conjugation by x is an automorphism c_x on W and can therefore be extended to $W_{l^n\times l^n}$ by letting it act on the matrix entries. Furthermore, we can read x as the diagonal matrix $M_x = x\cdot\mathbf{1}$ in $V_{l^n\times l^n}$. Then, the extension of c_x on $W_{l^n\times l^n}$ is the conjugation by this matrix M_x. This automorphism on $W_{l^n\times l^n}$ will be called C_x in the sequel and we remark that C_x acts on $F = F\cdot\mathbf{1} = F\otimes_E E$ as σ, with $\langle\sigma\rangle = G(F/E)$ as above.

On the other hand, $\sigma\otimes 1:\ F\otimes_E V\to F\otimes_E V$ is another automorphism on $W_{l^n\times l^n}$. As the restriction of $\sigma\otimes 1$ to $F\otimes_E E$ is by construction the old isomorphism σ, the actions of C_x and $\sigma\otimes 1$ coincide on $F = F\otimes_E E$.

Therefore, $C_x(\sigma\otimes 1)^{-1}$ is a central automorphism on $W_{l^n\times l^n}$, i.e. it acts trivially on the centre $Z(W_{l^n\times l^n}) = F\cdot\mathbf{1} = F$. The theorem of Skolem-Noether now implies that $C_x(\sigma\otimes 1)^{-1}$ is the conjugation C_w by some $w\in W_{l^n\times l^n}$, i.e.

$$\sigma\otimes 1 = C_w^{-1}C_x = C_{w^{-1}x}.$$

As $(\sigma\otimes 1)^{l^n} = \mathrm{id}$, we furthermore conclude $(C_{w^{-1}x})^{l^n} = \mathrm{id}$. This means that the conjugation by

$$(w^{-1}x)^{l^n} = (w^{-1})^{1+x^{-1}+\ldots+x^{-l^n+1}}x^{l^n}$$

is trivial on $W_{l^n\times l^n}$ and, as $x^{l^n}\in W$ (more precisely $x^{l^n}\in\mathbb{Q}(\zeta_i)\otimes_{\mathbb{Q}_l}\mathcal{Q}U_i$ has a component in W but we suppress this here for the sake of brevity), we conclude

$$(w^{-1}x)^{l^n}\in Z(W_{l^n\times l^n}) = F\cdot\mathbf{1} = F.$$

Finally, we choose

$$A = (F\otimes_E E/E\otimes_E E, \sigma\otimes 1 = C_{w^{-1}x}, (w^{-1}x)^{l^n}) = (F/E,\sigma,(w^{-1}x)^{l^n}).$$

By construction, we have $w\in W_{l^n\times l^n}$ and $x\in V$. Therefore,

$$w^{-1}x\in\bigoplus_{j=0}^{l^n-1}W_{l^n\times l^n}x^j = V_{l^n\times l^n}$$

and hence $A\subseteq V_{l^n\times l^n}$.

Lemma 7 *Let $A = (F/E,\sigma,(w^{-1}x)^{l^n})$ be as above. Then A splits, i.e. $A\sim E$ in $\mathrm{Br}(E)$.*

Proof: The cyclic algebra A splits if $(w^{-1}x)^{l^n}$ is a norm element in E, i.e. if there exists an element $f\in F$ with $\mathrm{N}_{F/E}(f) = (w^{-1}x)^{l^n}$, where $\mathrm{N}_{F/E} = \mathrm{N}_{\langle\sigma\rangle}$ is the Galois norm of the field extension F/E. To show this, we compute $(w^{-1}x)^{l^n}$ explicitely.

First, let k_x denote the conjugation by x on V, s.t. we can study the automorphism

$$\sigma\otimes k_x : F\otimes_E V\to F\otimes_E V.$$

By Lemma 6, we know

$$F \otimes_E V \cong W_{l^n \times l^n} = \mathrm{Hom}_W(V, V), \quad f \otimes v \mapsto l_f \circ r_v.$$

Here, we choose a basis $1, x, ..., x^{l^n - 1}$ of the free left W-vector space V. Then, we write $v = \sum_{i=0}^{l^n-1} w_i x^i$ and achieve that $l_f \circ r_v$ is represented by the matrix

$$\begin{pmatrix} fw_0 & fw_1 & \dots & fw_{l^n-1} \\ fw_{l^n-1}^{x^{-1}} x^{l^n} & fw_0^{x^{-1}} & \dots & . \\ \vdots & \vdots & \vdots & \vdots \\ fw_1^{x^{-(l^n-1)}} x^{l^n} & . & \dots & fw_0^{x^{-(l^n-1)}} \end{pmatrix}.$$

Here, we recall that we write the matrices from the right. Next,

$$(\sigma \otimes k_x)(f \otimes v) = \sigma(f) \otimes v^x \leftrightarrow \begin{pmatrix} \sigma(f) w_0^x & \sigma(f) w_1^x & \dots & \sigma(f) w_{l^n-1}^x \\ \sigma(f) w_{l^n-1} x^{l^n} & \sigma(f) w_0 & \dots & . \\ \vdots & \vdots & \vdots & \vdots \\ \sigma(f) w_1^{x^{-(l^n-2)}} x^{l^n} & . & \dots & . \end{pmatrix}.$$

A comparison of the two matrices shows that $\sigma \otimes k_x$ is the conjugation by x on $W_{l^n \times l^n}$, i.e.

$$\sigma \otimes k_x = C_x.$$

Next, we obtain

$$C_x \circ (\sigma \otimes 1)^{-1} = (\sigma \otimes k_x) \circ (\sigma \otimes 1)^{-1} = 1 \otimes k_x = C_{1 \otimes x} : \ F \otimes_E V \to F \otimes_E V.$$

Now, we see that

$$1 \otimes x = w \in W_{l^n \times l^n}$$

is the element s.t. $C_{w^{-1}x} = \sigma \otimes 1$.

Read as matrices in $V_{l^n \times l^n}$, we can write

$$w^{-1}x = (x^{-1}w)^{-1} = \left(\begin{pmatrix} x^{-1} & & & & \\ & x^{-1} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & x^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ x^{l^n} & 0 & 0 & \dots & 0 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} 0 & x^{-1} & 0 & \dots & 0 \\ 0 & 0 & x^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x^{-1} \\ x^{l^n-1} & 0 & 0 & \dots & 0 \end{pmatrix}^{-1}$$

We finally conclude that
$$(w^{-1}x)^{l^n} = 1$$
which is certainly a norm element. $\square$

Lemma 8 *Set $A = (F/E, \sigma, (w^{-1}x)^{l^n})$ and V as above. Then*
$$Z_{V_{l^n \times l^n}}(A) = (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}.$$

Proof: First, we have to read A in $V_{l^n \times l^n}$. For this, we observe that E and F are to be represented by the diagonal matrices $E = E \cdot \mathbf{1}$ and $F = F \cdot \mathbf{1}$. Then, choose a matrix $(v_{ij})_{i,j} \in Z_{V_{l^n \times l^n}}(A)$ and $f \cdot \mathbf{1} \in F \cdot \mathbf{1}$. We get
$$f^{-1}\mathbf{1}(v_{ij})f\mathbf{1} = (f^{-1}v_{ij}f) \overset{!}{=} (v_{ij}),$$
i.e. $f^{-1}v_{ij}f \overset{!}{=} v_{ij}$ for all $i, j = 0, ..., l^n - 1$. As this equation has to be fulfilled for all $f \in F$, but $v_{ij} = w_0 + ... + w_{l^n-1}x^{l^n-1} \in V$, we conclude that $v_{ij} = w_0 \in W$. Therefore, $Z_{V_{l^n \times l^n}}(A) \subseteq W_{l^n \times l^n}$.

Next, we conjugate by $w^{-1}x$ and see $Z_{V_{l^n \times l^n}}(A) \subseteq (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$.

It remains to show the other inclusion $(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \subseteq Z_{V_{l^n \times l^n}}(A)$. But this is obvious, because $A = \bigoplus_{j=0}^{l^n-1} F \cdot \mathbf{1}(w^{-1}x)^j$ and $v \in (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ commutes with $w^{-1}x$ as well as with $a \in F \cdot \mathbf{1} = Z(W_{l^n \times l^n})$. Thus, v commutes with $a_0 + ... + a_{l^n-1}(w^{-1}x)^{l^n-1} \in \bigoplus_{j=0}^{l^n-1} F(w^{-1}x)^j = A$, too. $\square$

Corollary 2 *$(W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ is a central simple $Z(A) = E$-algebra.*

Proof: This is true by the centralizer theorem. $\square$

Lemma 9 *With the above notations, we have*
$$V \cong (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}.$$
Moreover,
$$F \otimes_E (W_{l^n \times l^n})^{\langle w^{-1}x \rangle} \xrightarrow{\cong} W_{l^n \times l^n}, \quad f \otimes w \mapsto fw,$$
and
$$A \otimes_E Z_{V_{l^n \times l^n}}(A) \xrightarrow{\cong} V_{l^n \times l^n}, \quad a \otimes v \mapsto av,$$
are isomorphisms.

Proof: First, $V \sim (W_{l^n \times l^n})^{\langle w^{-1}x \rangle}$ in $\mathrm{Br}(E)$ by the centralizer theorem which states that
$$Z_{V_{l^n \times l^n}}(A) \sim A^{op} \otimes_E V_{l^n \times l^n}$$
in $\mathrm{Br}(E)$. By Lemma 7, we know that $A \sim E$ in $\mathrm{Br}(E)$. Because A^{op} is the inverse of A in $\mathrm{Br}(E)$, we conclude $A^{op} \sim E$ in $\mathrm{Br}(E)$, too. Thus,
$$(W_{l^n \times l^n})^{\langle w^{-1}x \rangle} = Z_{V_{l^n \times l^n}}(A) \sim A^{op} \otimes_E V_{l^n \times l^n} \sim E \otimes_E V_{l^n \times l^n} \sim V.$$

Next, we compute the respective degrees over E:

$$[(W_{l^n\times l^n})^{\langle w^{-1}x\rangle} : E] = [W_{l^n\times l^n} : E]/l^n = l^n[W : E] = l^n[W : F][F : E] = l^{2n}[W : F]$$

and

$$[V : E] = [V : W][W : F][F : E] = l^{2n}[W : F].$$

Thus, V and $(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ are as Brauer equivalent algebras of the same degree isomorphic.

We turn to the second isomorphism. As $Z_{V_{l^n\times l^n}}(A) = (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ is a central simple E-algebra, $F\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ is a central simple F-algebra. We compute the respective degrees over F:

$$[V_{l^n\times l^n} : E] = [V_{l^n\times l^n} : W_{l^n\times l^n}][W_{l^n\times l^n} : F][F : E] = l^{2n}[W_{l^n\times l^n} : F]$$

and, by the centralizer theorem,

$$\begin{aligned}[V_{l^n\times l^n} : E] &= [A : E][Z_{V_{l^n\times l^n}}(A) : E]\\ &= [F : E]^2[(W_{l^n\times l^n})^{\langle w^{-1}x\rangle} : E] = l^{2n}[(W_{l^n\times l^n})^{\langle w^{-1}x\rangle} : E]\end{aligned}$$

imply

$$\begin{aligned}[W_{l^n\times l^n} : F] &= [(W_{l^n\times l^n})^{\langle w^{-1}x\rangle} : E] = [(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}\otimes_E F : E\otimes_E F]\\ &= [(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}\otimes_E F : F].\end{aligned}$$

Next, $F\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}\to W_{l^n\times l^n}$, $f\otimes w\mapsto fw$, is injective because otherwise the kernel would form a non-trivial two-sided ideal. But $F\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ is a central simple F-algebra. Thus, the only non-trivial two-sided ideal is $F\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ itself, which is impossible because $E\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}\subseteq F\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ maps to $(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}\subseteq W_{l^n\times l^n}$ and thus the kernel can not be $F\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$. This implies $F\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}\subseteq W_{l^n\times l^n}$. As both sides are of the same degree over F, we conclude $F\otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle} = W_{l^n\times l^n}$.

Finally, we show $A\otimes_E Z_{V_{l^n\times l^n}}(A)\cong V_{l^n\times l^n}$. As A and $Z_{V_{l^n\times l^n}}(A)$ are central simple E-algebras, $A\otimes_E Z_{V_{l^n\times l^n}}(A)$ is a central simple E-algebra, too. We again show that the respective degrees over E coincide:

$$[V_{l^n\times l^n} : E] = [A : E][Z_{V_{l^n\times l^n}}(A) : E] = [A\otimes_E Z_{V_{l^n\times l^n}}(A) : E].$$

Next, the homomorphism $A\otimes_E Z_{V_{l^n\times l^n}}(A)\to V_{l^n\times l^n}$, $a\otimes v\mapsto av$, again is injective. Dimension comparison implies $A\otimes_E Z_{V_{l^n\times l^n}}(A) = V_{l^n\times l^n}$. □

Proposition 4 *With the above notations, assume that $W^x = W$ and moreover $SK_1(\mathbb{Q}_l(\zeta_i)\otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) = 1$ for all i. Then*

$$SK_1(\tilde{W}) = SK_1((W_{l^n\times l^n})^{\langle w^{-1}x\rangle}) = 1.$$

Proof: We are still in the case $V = \tilde{W}$. With $V_{l^n\times l^n} = A \otimes_E Z_{V_{l^n\times l^n}}(A)$, it therefore suffices to compute

$$\begin{aligned} SK_1(V) = SK_1(V_{l^n\times l^n}) &= SK_1(A \otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle}) \\ &\stackrel{1}{=} SK_1(A) \times SK_1((W_{l^n\times l^n})^{\langle w^{-1}x\rangle}) \\ &\stackrel{2}{=} 1 \times SK_1((W_{l^n\times l^n})^{\langle w^{-1}x\rangle}). \end{aligned}$$

in the sequel. For $\stackrel{2}{=}$, we use that A splits and therefore $SK_1(A) = 1$; moreover, A and $(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ have coprime Schur indices which implies $\stackrel{1}{=}$ by [2, Lem 5, p. 160].

Now, we choose a $v \in V_{l^n\times l^n}$ with $\mathrm{nr}_{V_{l^n\times l^n}/E}(v) = 1$. It represents an element in $SK_1(V_{l^n\times l^n})$. By the above, the class of v can be read as

$$[v] = (1, [\tilde{v}]) = [1 \otimes \tilde{v}] = [\tilde{v}]$$

with $\tilde{v} \in (W_{l^n\times l^n})^{\langle w^{-1}x\rangle} \subseteq V_{l^n\times l^n}$ and

$$\mathrm{nr}_{(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}/E}(\tilde{v}) = 1.$$

Therefore, v and $\tilde{v}$ only differ by a factor in $[(V_{l^n\times l^n})^\times, (V_{l^n\times l^n})^\times]$. It hence suffices to show that $\tilde{v} \in [(V_{l^n\times l^n})^\times, (V_{l^n\times l^n})^\times]$ for $SK_1(V_{l^n\times l^n}) = 1$.

For the computation of $\mathrm{nr}_{(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}/E}$, let M be a splitting field of $(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ with $M \supseteq F$. Thus, as

$$(W_{l^n\times l^n})^{\langle w^{-1}x\rangle} \subseteq F \otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle} = W_{l^n\times l^n},$$

we get

$$M_{m\times m} = M \otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle} = M \otimes_F F \otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle} = M \otimes_F W_{l^n\times l^n}$$

for a certain $m \in \mathbb{N}$, i.e. M is also a splitting field of $W_{l^n\times l^n}$. This implies

$$1 = \mathrm{nr}_{(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}/E}(\tilde{v}) \stackrel{1}{=} \mathrm{nr}_{W_{l^n\times l^n}/F}(\tilde{v}),$$

where $\stackrel{1}{=}$ holds due to the isomorphism $F \otimes_E (W_{l^n\times l^n})^{\langle w^{-1}x\rangle} = W_{l^n\times l^n}$, $1 \otimes \tilde{v} \mapsto 1 \cdot \tilde{v}$ and the common splitting field $M \supseteq F \supseteq E$ of $(W_{l^n\times l^n})^{\langle w^{-1}x\rangle}$ and $W_{l^n\times l^n}$. But, by assumption, $SK_1(W_{l^n\times l^n}) = SK_1(W) = 1$ and hence

$$\tilde{v} \in [(W_{l^n\times l^n})^\times, (W_{l^n\times l^n})^\times] \subseteq [(V_{l^n\times l^n})^\times, (V_{l^n\times l^n})^\times].$$

This concludes the proof. $\square$

3.1.3 The intermediate case $0 < d < n$

Finally, the triviality of $SK_1(\tilde{W})$ in the intermediate cases for $0 < j < n$ is a consequence of the extreme cases: We fix a $0 < d < n$ and set $y := x^{l^d}$ and $m := n - d$. Thus,

$$\begin{aligned} V &= W \oplus Wx^{l^d} \oplus ... \oplus Wx^{l^d(l^{n-d}-1)} \\ &= W \oplus Wy \oplus ... \oplus Wy^{(l^{n-d}-1)} = \bigoplus_{j=0}^{l^m-1} Wy^j. \end{aligned}$$

As $\tilde{W} = V_{l^d \times l^d}$, it suffices to compute $SK_1(V)$. But V is now of the same form as $\tilde{W}$ in the case $d = 0$, with x replaced by y and n replaced by m. Thus, we only have to check that the above arguments apply to this V in the same manner. As it can be seen easily that we can copy the above literally, we leave this to the reader.

Hence, we have seen that $SK_1(\tilde{W}) = 1$ for every Wedderburn component of $\mathcal{Q}G$. This concludes the proof of Theorem 1. □

3.2 $\mathbb{Q}_l$-q-elementary groups G

Next, we consider the case of $\mathbb{Q}_l$-q-elementary groups G with $q \neq l$. Here, our result on the triviality of the reduced Whitehead group is stronger than in the case $q = l$ because it holds without assumptions:

Theorem 2 *Let G be a $\mathbb{Q}_l$-q-elementary group with $q \neq l$ prime. Then*

$$SK_1(\mathcal{Q}G) = 1.$$

The proof of this theorem closely follows the proof of Theorem 1. Thus, we only give a short outline how to adapt the ideas used for the case $q = l$ to our new situation.

To do so, we first recall that the $\mathbb{Q}_l$-q-elementary group G is a direct product $G = H \times \Gamma$ with H a finite $\mathbb{Q}_l$-q-elementary group. More precisely, $H = \langle s \rangle \rtimes H_q$ with $\langle s \rangle$ a cyclic group of order prime to q and a q-group H_q whose action on $\langle s \rangle$ induces a homomorphism $H_q \to G(\mathbb{Q}_l(\zeta)/\mathbb{Q}_l)$ for ζ a primitive root of unity of order $|\langle s \rangle|$.

We now take $\langle s_l \rangle$ the l-Sylow subgroup of $\langle s \rangle$, thus $\langle s \rangle = \langle s_l \rangle \times \langle s' \rangle$, and obtain $G = \langle s_l \rangle \rtimes U$ with $U = (\langle s' \rangle \rtimes H_q) \times \Gamma$ and $\langle s' \rangle \rtimes H_q$ an l-prime group. Still, U acts on $\langle s_l \rangle$ via Galois automorphisms.

As in the $\mathbb{Q}_l$-l-elementary case, we fix a finite set $\{\beta_i\}$ of representatives of the $G(\mathbb{Q}_l^c/\mathbb{Q}_l)$-orbits of the irreducible $\mathbb{Q}_l^c$-characters of $\langle s_l \rangle$. Let also ζ_i denote a fixed primitive root of unity with $\beta_i(s) = \zeta_i$. This time, $\mathbb{Q}_l(\zeta_i)$ is not unramified because $\langle s_l \rangle$ is an l-group, but $\mathbb{Q}_l(\zeta_i)/\mathbb{Q}_l$ remains cyclic because l is odd. Let again $U_i := \{u \in U : \beta_i^u = \beta_i\}$ denote the stabilizer group of β_i. Thus, $U_i \lhd U$ and $A_i := U/U_i \leq G(\mathbb{Q}_l(\beta_i)/\mathbb{Q}_l) = G(\mathbb{Q}_l(\zeta_i)/\mathbb{Q}_l)$ is cyclic because $G(\mathbb{Q}_l(\zeta_i)/\mathbb{Q}_l)$ is. We fix a representative $x \in U$ with $\langle \overline{x} \rangle = U/U_i = A_i$. Then, $\overline{x}$ maps to some τ under the

injection $U/U_i \rightarrowtail G(\mathbb{Q}_l(\zeta_i)/\mathbb{Q}_l)$; and $|x| = |U/U_i| =: l^n$ with again x, τ and n depending on i. Finally, we set $G_i := \langle s_l \rangle \rtimes U_i$.

Now, we may compute the isomorphism

$$\mathcal{Q}G = \bigoplus_i e_i \mathcal{Q}G \cong \bigoplus_i (\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle.$$

In this case that $q \neq l$, the group U_i is not pro-l and therefore the structure of $\mathcal{Q}U_i$ differs from the structure of the analogous object in the pro-l case as stated in Section 2. Yet, we may collect all relevant information on $\mathcal{Q}U_i$ easily. Recall that U is the direct product $U = (\langle s' \rangle \rtimes H_q) \times \Gamma$ and therefore U_i also is the direct product $U_i = H' \times \Gamma$ with H' a subgroup of $\langle s' \rangle \rtimes H_q$. Now, [1, 74.11, p. 740] implies that $\mathcal{Q}U_i$ is the direct sum of matrix rings over the fields $F' = \mathbb{Q}_l(\eta') \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma$ for certain characters η' of H'. Because F' and $\mathbb{Q}_l(\zeta_i)$ are linearly disjoint over $\mathbb{Q}_l$, the algebra $\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i$ also is a direct sum of matrix rings over fields $F = \mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} F'$. This proves

Proposition 5 *With the above notations, we have*

$$SK_1(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}(U_i)) = 1.$$

This proposition allows us to formulate Theorem 2 in the stronger form without any assumptions on $SK_1(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}(U_i))$.

It now remains to examine the semisimple algebra $(\mathbb{Q}_l(\zeta_i) \otimes_{\mathbb{Q}_l} \mathcal{Q}U_i) \star \langle x \rangle$. We may transfer the computation for the case $q = l$ directly to our new situation.

This concludes the proof of Theorem 2. $\square$

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